INVARIANT IMBEDDING AND WAVE PROPAGATION IN STOCHASTIC MEDIA

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Summary

In a series of recent papers the principle of invariant imbedding, which represents an extension of the invariance principles of Ambarzumian and Chandrasekhar, has been employed in the study of a variety of physical processes including radiative transfer, neutron transport, random walk and scattering, and wave propagation.

Use of this principle leads to the formulation of various functional equations describing the process under consideration. Redheffer has used a similar approach in the treatment of several problems of electromagnetic theory.

The aim of the present paper is to extend previous results and techniques to cover cases involving plane wave propagation in stochastic media.

In general terms, the approach involves first the derivation of <u>stochastic</u> functional equations for reflection and transmission coefficients, followed by the taking of expected values of appropriate functions of the random state variables. This makes possible the determination of their characteristic functions and distribution functions, by means of still other functional equations, or by computational schemes of the Monte Carlo type.

We discuss the particular example in which a plane wave is incident on a stratified slab which is characterized by stochastic wave numbers in each stratum. The distribution

functions for the amplitude of the random reflected and transmitted waves are then determined as functions of the thickness of the slab. The effects of multiple scattering are taken into account.

The emphasis throughout is upon the general applicability of these functional equation procedures to problems of propagation in stochastic media, using methods quite distinct from the classical techniques.

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1. Introduction

In this paper, we wish to describe an application of the theory of invariant imbedding, [4 - 11], to the propagation of electromagnetic waves through an inhomogeneous medium, especially where the inhomogeneity is stochastic in origin. Problems of this sort arise in a variety of fields including radio wave propagation and acoustics.

There have been a large number of papers devoted to this topic, most using the routine perturbation approach, and a few the concept of a stratified medium; cf. Redheffer, [22], Luneberg, [19], Bremmer, [13], Tatarskii, [24], Gronwall, [16], et al*. Topics of current interest in radio wave propagation are discussed in [12,14,25,27].

Our aim is first of all to illustrate the applicability of functional equations and principles of invariance to the study of various types of wave propagation. The basic ideas were sketched in our note, [7], and fuller mathematical details will be given in papers shortly to appear. Secondly, we wish to point out the importance of functional equations in stochastic variables, prior to the appearance of any

*In his book on Radiative Transfer, Chandrasekhar refers to some earlier uses of invariance principles by Stokes and Rayleigh.

expected values. These relations can be used, as discussed below, to furnish numerical solutions along Monte Carlo lines. Finally, we wish to emphasize that the nonlinear aspects of the recurrence relations that are derived make it important not to use expected values, but rather to examine the actual probability distribution of the random variables that appear.

2. The Physical Process

We wish to consider the problem of determining the characteristics of waves reflected from a randomly inhomogeneous medium and the properties of waves transmitted through such a medium. In particular, we limit ourselves to the case in which a plane scalar wave is normally incident on a slab bounded by parallel planes, the wave number or index of refraction of which is a random function of distance from an interface. We assume that the media in which the slab is contained have constant wave numbers and, as usual, that time variations are simple harmonic. Ultimately we are concerned with random solutions of the reduced wave equation in one dimension

$$(1) \qquad w''(x) + k^2(x)w(x) = 0,$$

where the local wave number, $k(x)$, is a random function of x. The index of refraction, $n(x)$, is expressable as

$$(2) \qquad n(x) = \frac{k(x)}{k_0}.$$

Since it is the method of attack which we wish to emphasize, before taking up the stochastic case, we solve the reflection and transmission problems for the case in which $k(x)$ is deterministic, making use of the notion of invariant imbedding and a wave-localization principle which we first gave in [7]. Then we take up the problem of reflections from, and transmissions through, a medium which has random stratifications, each stratum being bounded by planes perpendicular to the direction of propagation of the incident wave and each having a random wave number. For convenience, we assume that these random wave numbers are independent, but no assumptions are made concerning "smallness" of departures of the wave numbers from their average values.

Some generalizations and indications of future work are provided in the discussion of §6.

3. Deterministic Wave Propagation

To illustrate some of the ideas which we shall employ in the treatment of the stochastic problem, we first discuss the problem of the reflection of a plane wave normally incident on an inhomogeneous slab, a problem of some difficulty and importance in itself, [19]. Consider a plane wave, $\exp i(k_0x - \omega t)$, arriving from the homogeneous space, $x < 0$, and approaching the inhomogeneous slab, $0 < x < b$. The space $x > b$ is assumed to be homogeneous.

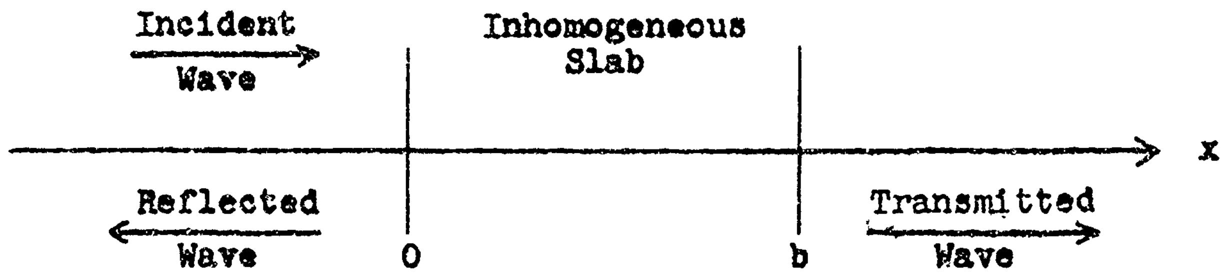

Fig. 1

A Plane Wave Incident on an Inhomogeneous Slab

The medium $x < 0$ is characterized by the wave number k_0, and the medium $x > b$ by the wave number k_1. The inhomogeneous slab, $0 < x < b$, is characterized by the wave number $k(x)$, which is assumed to be sectionally smooth. Our objective is the determination of the reflected wave, Ae^{-ik_0x}, where here, and in what follows, we shall suppress the time factor $e^{-i\omega t}$.

We first imbed this problem within a class of problems in which the incident wave and the inhomogeneous slab are suitably altered. More precisely, we consider the problem of determining the wave reflected in the half-space $x \leq z$, where $0 \leq z \leq b$, as a result of having the wave $\exp\left[ik(z - 0)(x - z)\right]$ incident on the inhomogeneous slab $z \leq x \leq b$. As usual, $k(z - 0)$ denotes the limit of $k(x)$ as x approaches z from below and $k(z + 0)$ is the limit as x approaches z from above. As before, we take the space $b < x$ to be homogeneous with the wave number k_1.

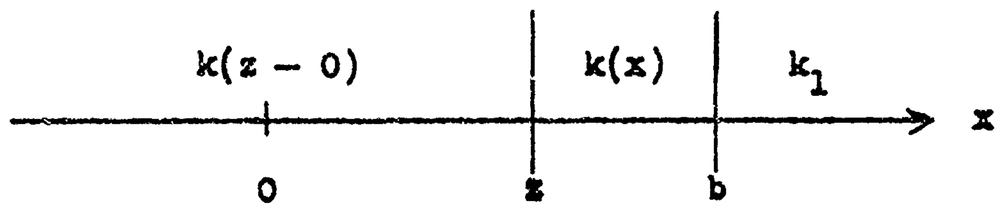

Fig. 2
The Geometry of the Class of Slab Problems

For the modified problem the space $x < z$ has the constant wave number $k(z - 0)$, and we denote the reflected wave by $u(z) \exp\left[- ik(z - 0)(x - z)\right]$, where the explicit dependence of the coefficient of reflection, $u(z)$, on the position of the left face of the slab, z, is indicated.

Observe that when $z = 0$ we are confronted with the original problem, and when $z = b$ we are confronted with the simple problem of determining the reflection coefficient for waves incident on the plane interface of two homogeneous media. In the latter case we have

$$(1) \qquad u(b) = \frac{k(b - 0) - k_1}{k(b - 0) + k_1},$$

which follows from the usual assumptions that are made concerning the continuity at the interface of the wave function and its first derivative with respect to x.

We now wish to show that a knowledge of the reflection coefficient for the inhomogeneous slab's extending from $z + \Delta$ to b enables one to find the reflection coefficient for the slab $[z,b]$. This is accomplished through use of the

localization principle given in [7], valid under mild restrictions on the function $k(x)$, which in turn was suggested by Bremmer's paper, [13]. Let us now indicate this principle.

Return for a moment to the original problem with $z = 0$ and consider a plane wave $\exp(ik_0x)$ arriving from the homogeneous space $x < 0$ and entering the inhomogeneous slab. At the boundary the incident wave is split into a reflected wave and a refracted wave. There is an immediate reflected wave and an immediate refracted wave obtained by supposing that the inhomogeneous medium is actually homogeneous with wave number $k(+0)$. If the inhomogeneous medium is now taken to be the limit of a sequence of interfaces with this type of reflection and refraction occurring at each interface, we can obtain the total reflected wave and the disturbance within the inhomogeneous slab by adding up the effects of the reflected and refracted waves obtained in this way and then passing to the limit. This principle is valid regardless of the position of the left face of the slab, z, $0 \leq z \leq b$.

To apply this principle, we consider the situation illustrated in Fig. 3, in which the wave numbers of the strata are as shown.

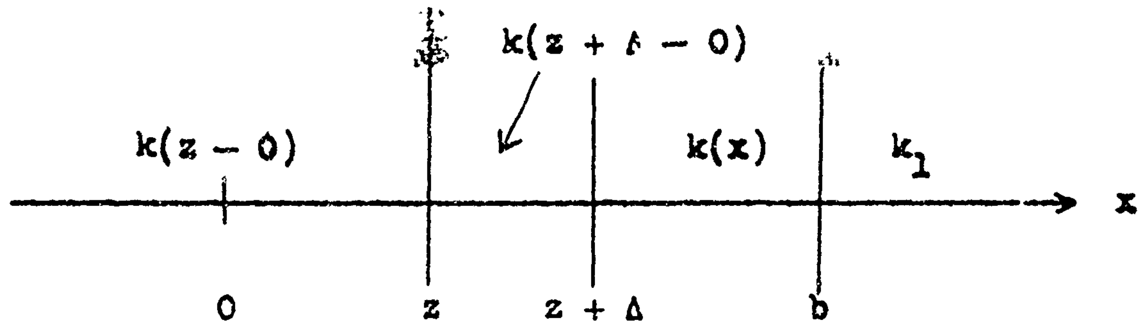

Fig. 3
The Geometry for a Class of Modified Problems

To within terms of orders zero and one, the reflection coefficient for the slab $[z, b]$ is given by the sum of three terms. The first is the immediately reflected wave. The second is the wave that arises from transmission of the incident wave through the interface $x = z$, reflection at the interface $x = z + \Delta$ and transmission through the interface $x = z$. The third component arises from transmission of the incident wave through the interface at $x = z$, reflection at $x = z + \Delta$, reflection at $x = z$, reflection at $x = z + \Delta$, and finally transmission through the interface at $x = z$.

Mathematically, we are led to the equation

$$(2) \qquad u(z) = \frac{k(z-0) - k(z+\Delta-0)}{k(z-0) + k(z+\Delta-0)}$$

$$+ \frac{2k(z-0)}{k(z-0) + k(z+\Delta-0)} u(z+\Delta) e^{2ik(z+\Delta-0)\Delta} \frac{2k(z+\Delta-0)}{k(z-0) + k(z+\Delta-0)}$$

$$+ \frac{2k(z-0)}{k(z-0) + k(z+\Delta-0)} e^{ik(z+\Delta-0)\Delta} u(z+\Delta) e^{ik(z+\Delta-0)\Delta}$$

$$\cdot \frac{k(z+\Delta-0) - k(z-0)}{k(z-0) + k(z+\Delta-0)} e^{ik(z+\Delta-0)\Delta} u(z+\Delta) e^{ik(z+\Delta-0)\Delta}$$

$$\cdot \frac{2k(z+\Delta-0)}{k(z+\Delta-0) + k(z-0)} + o(\Delta).$$

A passage to the limit in which Δ tends to zero then shows that $u(z)$ satisfies the Riccati equation

$$(3) \qquad u'(z) = \frac{k'}{2k} - 2iku - \frac{k'}{2k} u^2,$$

where the primes denote differentiation with respect to z.

Integration of this differential equation, which is easily accomplished numerically on a high speed digital computer or by hand, using the end condition of equation (1), on the interval $[0,b]$ then yields the desired reflection coefficient, $u(0)$.

Similar Riccati equations for the impedance and reflection coefficient of nonuniform transmission lines have been given by Pierce, [21], and by Walker and Wax, [26]; see also the discussion given by Schelkunoff, [23], and Osterberg, [20]. The idea of concentrating attention on the

reflection coefficient as a function of the thickness of the slab goes back to Ambarzumian, [1], who applied it in certain radiative transfer problems involving diffuse reflection from a foggy medium. The notion was considerably extended and developed by Chandrasekhar, [15] and Bellman and Kalaba, [4]. In addition it forms the basis for the treatment by Bellman, Kalaba, and Wing, [8 - 11] of various problems of neutron multiplication in fissionable material, including the determination of critical mass.

The point here is that it is easier to deal with the nonlinear Riccati equation and the end condition, insofar as numerical computation is concerned, than it is to solve the linear two-point boundary value problem for the wave within the slab. Furthermore, in many cases, it is the reflected wave rather than the wave within the slab which is of primary physical importance. Once the reflection coefficient $u(z)$ has been determined, though, for $0 \leq z \leq b$, the determination of the wave within the slab is also reduced to an initial value problem.

Similar considerations enable us to determine the transmission coefficient, $v(z)$. For $z = b$ the problem reduces to that of determining the transmission through the interface between two homogeneous media having wave numbers $k(b - 0)$ and k_1. We have the usual end condition

$$(4) \qquad v(b) = \frac{2k(b - 0)}{k(b - 0) + k_1} .$$

To calculate the transmission coefficient for the case in which the inhomogeneous slab extends from z to b, from a knowledge of the transmission coefficient for the case in which the slab extends from $z+\Delta$ to b, we write the following equation:

$$(5)\quad v(z) = \frac{2k(z-0)}{k(z-0)+k(z+\Delta-0)} e^{ik(z+\Delta-0)\Delta}\Big\{v(z+\Delta) + u(z+\Delta)e^{ik(z+\Delta-0)\Delta} \cdot \frac{k(z+\Delta-0)-k(z-0)}{k(z+\Delta-0)+k(z-0)} e^{ik(z+\Delta-0)\Delta}\, v(z+\Delta)\Big\} + o(\Delta).$$

It states that excluding terms of order higher than the first in Δ the total transmitted wave consists of one component which is transmitted through the interface $x = z$, is transmitted through the interface $x = z + \Delta$, and is transmitted through the interface $x = b$, and a second component which arises from transmission through $x = z$, reflection at $x = z + \Delta$, reflection at $x = z$ and transmissions at $x = z + \Delta$ and $x = b$. A passage to the limit in which Δ tends toward zero then yields the differential equation

$$(6)\qquad v' = \left(\frac{k'}{2k} - u\frac{k'}{2k} - ik\right)v.$$

Assuming that $u(z)$ has been calculated on the interval $0 \leq z \leq b$, we see that the equations (4) and (6) permit the determination of $v(z)$ on the same interval; in particular the transmission coefficient for the inhomogeneous slab, $v(0)$, can be calculated.

We wish to point out the strong similarity between

equations (3) and (6) above which describe reflection and transmission of waves and the equations in (6.2) of our previous paper, [8], which describe reflection and transmission of neutrons in a rod of fissionable material. Abstractly the derivations of the equations for the wave and the particle cases are identical. We plan in the future to investigate the implications of this wave-particle analogy in more detail.

4. Stochastic Wave Propagation

Let us now turn our attention to the case in which a wave is normally incident on a stratified slab, each stratum of which is characterized by a wave number which is a random variable. Our objectives will be the characterization of the stochastic reflected and refracted waves.

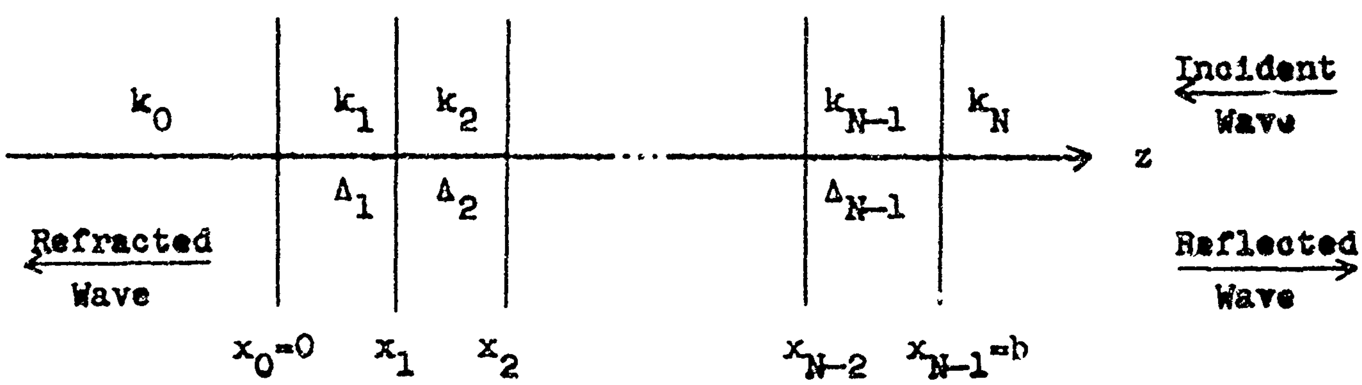

Fig. 4
A Stochastic Stratified Slab of Thickness b

Referring to the above figure, we denote the wave number of the medium $z < 0$ by k_0, and that of the stratum between

x_m and x_{m+1} by the random variable k_{m+1}, $m = 0,1,\ldots,N-2$. For $z > x_{N-1} = b$, the medium has wave number k_N. Thus the slab $0 < z < x_{N-1}$ is divided into $N - 1$ strata. We put $\Delta_m = x_m - x_{m-1}$. The incident wave, which arrives at the interface $z = x_{N-1}$ from $+\infty$, is specified by the function $e^{ik_N(z-x_{N-1})}$, where now the time factor $e^{i\omega t}$ is suppressed. The reflected wave is denoted by $R_N e^{-ik_N(z-x_{N-1})}$, $z > x_{N-1}$, and the refracted wave is denoted by $T_N e^{ik_0 z}$, $z < 0$.

Our attack on the problem of the determination of the random variables R_N and T_N will be along the following lines. We first imbed the problem for a fixed value of N within the class of problems for which $N = 1,2,\ldots$. Next, using the localization principle, we derive a relationship between the random variables R_N and R_{N-1}. Lastly, using analytic, Monte Carlo, or perturbation techniques we determine successively the distribution functions for the reflection coefficient, R_N, and the transmission coefficient, T_N. In this paper we assume that the random wave numbers k_m are independent, but no assumptions of smallness of variation of the random numbers k_m about their means need be made.

The simple case in which $N = 1$ is easily treated. The incoming wave is $e^{ik_1 z}$, $z > 0$, the reflected wave is $R_1 e^{-ik_1 z}$, $z > 0$, and the refracted wave is $T_1 e^{ik_0 z}$, $z < 0$. In terms of the wave numbers k_0 and k_1, the reflection and transmission coefficients are given by

(1) $$R_1 = \frac{k_1 - k_0}{k_1 + k_0},$$

(2) $$T_1 = \frac{2k_1}{k_1 + k_0}.$$

These relations follow from the assumed continuity of the wave function and its first derivative at the interface $z = 0$.

We now turn to the case illustrated in Fig. 4. The total reflected wave is composed of an immediately reflected wave, a wave which arises from transmission through the interface at $z = x_{N-1}$, reflection at the face $z = x_{N-2}$, and retransmission through the interface at $z = x_{N-1}$, and so on. Expressed mathematically, this statement becomes

(3) $$R_N = \frac{k_N - k_{N-1}}{k_N + k_{N-1}} + \frac{2k_N}{k_N + k_{N-1}} e^{ik_{N-1}(-\Delta_{N-1})} R_{N-1} e^{-ik_{N-1}\Delta_{N-1}} \cdot \frac{2k_{N-1}}{k_{N-1} + k_N} + \cdots,$$

where the dots denote the contributions from terms involving more than one reflection from the medium $z < x_{N-2}$. To evaluate these terms we note that each component incident on the interface $z = x_{N-1}$ from the left gives rise to a reflected wave which is then partially reflected at the interface $z = x_{N-2}$ and is then partially transmitted through the interface $z = x_{N-1}$. These components are the terms of a geometric series with common ratio

$$\frac{k_{N-1} - k_N}{k_{N-1} + k_N} R_{N-1} e^{-2ik_{N-1}\Delta_{N-1}}.$$

Equation (3) becomes

$$(4) \qquad R_N = \frac{k_N - k_{N-1}}{k_N + k_{N-1}} + \left(\frac{4k_N k_{N-1}}{(k_N+k_{N-1})^2}\, e^{-2ik_{N-1}\Delta_{N-1}} R_{N-1}\right) \cdot \left(\frac{1}{1 - \frac{k_{N-1} - k_N}{k_{N-1} + k_N}\, e^{-2ik_{N-1}\Delta_{N-1}} R_{N-1}}\right) .$$

It is convenient to introduce the local reflection and transmission coefficients for waves incident from the right on the interface $x = x_{N-1}$:

$$(5) \qquad r_N = \frac{k_N - k_{N-1}}{k_N + k_{N-1}} ,$$

$$(6) \qquad t_N = \frac{2k_N}{k_N + k_{N-1}} ,$$

and the corresponding coefficients for waves incident from the left,

$$(7) \qquad r_N' = \frac{k_{N-1} - k_N}{k_N + k_{N-1}} ,$$

$$(8) \qquad t_N' = \frac{2k_{N-1}}{k_N + k_{N-1}} .$$

These coefficients obviously satisfy Stokes' relations,

$$(9) \qquad r_N + r_N' = 0,$$

$$(10) \qquad t_N t_N' = 1 - r_N^2,$$

$$(11) \qquad t_N + t_N' = 2,$$

so that equation (4) becomes

$$(12)\qquad R_N = r_N + t_N t_N' e^{-2ik_{N-1}\Delta_{N-1}} R_{N-1} \frac{1}{1 - r_N' e^{-2ik_{N-1}\Delta_{N-1}} R_{N-1}}$$

$$= \frac{r_N + e^{-2ik_{N-1}\Delta_{N-1}}(t_N t_N' - r_N r_N')R_{N-1}}{1 - r_N' e^{-2ik_{N-1}\Delta_{N-1}} R_{N-1}} .$$

If, finally, we make use of Stokes' relations and introduce

$$(13)\qquad \delta_{N-1} = e^{-2ik_{N-1}\Delta_{N-1}},$$

we find

$$(14)\qquad R_N = \frac{r_N + \delta_{N-1}R_{N-1}}{1 + r_N R_{N-1}}, \quad N = 2,3,\ldots .$$

Equation (1) yields

$$(15)\qquad R_1 = r_1.$$

Equations (14) and (15) constitute the desired stochastic functional equations for the reflection coefficients R_N.

Similar considerations for the transmission coefficients yield the relations

$$(16)\qquad T_N = \frac{t_N e^{-ik_{N-1}\Delta_{N-1}}}{1 + r_N \delta_{N-1} R_{N-1}} T_{N-1}, \quad N = 2,3,\ldots,$$

$$(17)\qquad T_1 = t_1.$$

5. Applications of the Preceding Results

We now consider the uses to which we may put the stochastic functional equations derived in the last section. The basic problem that confronts us is that of determining the distribution of R_N and T_N, given the distribution of the random variables r_i and δ_i, $i = 1,2,\ldots,N-1$. Here, in speaking of the determination, we are primarily interested in the numerical determination.

We shall discuss three possible uses of the preceding results.

1. Monte Carlo. The formula seems ideally suited for a direct determination of the distribution for R_N and T_N using a Monte Carlo technique in conjunction with a digital computer which generates the random sequences $\{r_i\}$ and $\{\delta_i\}$. This will be easy to carry out even when the distributions of r_i and δ_i depend upon those of r_{i-1} and δ_{i-1}. Thousands of runs can be carried out in a matter of minutes.

2. Recurrence Relations for Distribution Functions. Since R_N is complex, we take as a fundamental distribution associated with this stochastic variable, the function

(1) $p_N(z,r)$ = the probability that R_N be in a circle about z with radius r.

Since the transformation in (4.14) maps the interiors of circles into the interiors of circles, it is clear that we

can obtain a relation for $p_N(z,r)$ in terms of the corresponding function for $N - 1$, [17].

Since, however, $p_N(z,r)$ depends upon the three real variables x, y and r, where $z = x + iy$, we see that this relation is not ideally suited for direct numerical computation.

Having obtained the formula, we can now introduce perturbation techniques. Assuming that r_i and δ_i may be written in the form

$$\text{(2)} \qquad \begin{aligned} r_i &= r + \epsilon\rho_i, \\ \delta_i &= d + \epsilon\Delta_i, \end{aligned}$$

where ϵ is a small parameter and ρ_i and Δ_i are random variables with known distributions, we can write

$$\text{(3)} \qquad p_N(z,r) = p_N^{(0)} + \epsilon p_N^{(1)} + \epsilon^2 p_N^{(2)} + \cdots,$$

and use the recurrence relations to obtain relations for $p_N^{(0)}$, $p_N^{(1)}$, ...

3. Stochastic Iteration. The formula in (4.14) yields a relation of the type

$$\text{(4)} \qquad R_N = \frac{u_N + v_N R_0}{w_N + z_N R_0},$$

where u_N, v_N, w_N, z_N are random variables determined by the sequences $\{r_i\}$ and $\{\delta_i\}$.

As a matter of fact, we know that (4.14) is equivalent to the matrix relation

$$(5)\qquad \begin{pmatrix} u_N & v_N \\ w_N & z_N \end{pmatrix} = \begin{pmatrix} r_N & \delta_{N-1} \\ 1 & r_N \end{pmatrix} \begin{pmatrix} u_{N-1} & v_{N-1} \\ w_{N-1} & z_{N-1} \end{pmatrix}.$$

These relations can now be used to determine the individual and joint moments of the u_N, v_N, w_N, z_N, in terms of the moments of the r_1 and δ_1, cf. [2].

Results of this type are useful in connection with the inverse problem of determining the distribution of the r_1 and δ_1, given the observed stochastic values of the R_N.

Each of these methods will have their analogues in the multi-dimensional case. From the standpoint of computational ease and effectiveness, we feel that the Monte Carlo method is the most promising.

6. Discussion

It is clear that this paper is but a first step toward a more comprehensive theory of wave propagation in stochastic media based on the principle of invariant imbedding, a radical departure from current practice. In future papers, we plan to discuss applications to the case of oblique incidence, other geometries (see [5,10]), and electromagnetic wave propagation, *per se*. In addition we propose to exploit the wave-particle similarity mentioned in §3.

Finally, let us mention other techniques for treating

nonlinear equations with stochastic elements, given in Bellman, [3], and Kalaba, [18], and which are relevant in wave propagation studies.

References

1. Ambarzumian, V. A., "Diffuse reflection of light by a foggy medium," Compt. Rend. Acad. Sci. URSS, vol. 38, 1943, pp. 229-232.

2. Bellman, R., "Limit theorems for non-commutative operations, I," Duke Math. J., vol. 21, 1954, pp. 491-500.

3. Bellman, R., "On the representation of the solution of a class of stochastic differential equations," Proc. Amer. Math. Soc., vol. 9, 1958, pp. 326-327.

4. Bellman, R., and R. Kalaba, "On the principle of invariant imbedding and propagation through inhomogeneous media," Proc. Nat. Acad. Sci. USA, vol. 42, 1956, pp. 629-632.

5. Bellman, R., and R. Kalaba, "On the principle of invariant imbedding and diffuse reflection from cylindrical regions," Proc. Nat. Acad. Sci. USA, vol. 43, 1957, pp. 514-517.

6. Bellman, R., and R. Kalaba, "Random walk, scattering, and invariant imbedding—I: one-dimensional case," Proc. Nat. Acad. Sci. USA, vol. 43, 1957, pp. 930-933.

7. Bellman, R., and R. Kalaba, "Invariant imbedding, wave propagation and the WKB approximation," Proc. Nat. Acad. Sci. USA, vol. 44, 1958, pp. 317-319.

8. Bellman, R., R. Kalaba, and G. M. Wing, "On the principle of invariant imbedding and one-dimensional neutron multiplication," Proc. Nat. Acad. Sci. USA, vol. 43, 1957, pp. 517-520.

9. Bellman, R., R. Kalaba, and G. M. Wing, "On the principle of invariant imbedding and neutron transport theory—I: one-dimensional case," J. of Math. and Mech., vol. 7, 1958, pp. 149-162.

10. Bellman, R., R. Kalaba, and G. M. Wing, Invariant imbedding and generalized transport theory—a basic stochastic functional equation, The RAND Corporation, Paper P-1390, June 3, 1958.

11. Bellman, R., R. Kalaba, and G. M. Wing, "Invariant imbedding and neutron transport theory—II: functional equations," J. of Math. and Mech., to appear.

12. Booker, H. G., and W. E. Gordon, "A theory of radio scattering in the troposphere," Proc. IRE, vol. 38, 1950, pp. 401-412.

13. Bremmer, H., "The WKB approximation as the first term of a geometric-optical series," The Theory of Electromagnetic Waves, a Symposium, Interscience Publishers, Inc., New York, 1951, pp. 105-115.

14. Carroll, T. J., and R. M. Ring, "Propagation of short radio waves in a normally stratified troposphere," Proc. IRE, vol. 43, 1955, pp. 1384-1390.

15. Chandrasekhar, S., Radiative Transfer, Oxford, 1950.

16. Gronwall, T. H., "Reflection of radiation from a finite number of equally spaced parallel planes," Phys. Rev., vol. 27, 1926, pp. 277-285.

17. Hurwitz, A., and R. Courant, Allgemeine Funktionentheorie und Elliptische Funktionen, Geometrische Funktionentheorie (in one volume), distributed by Interscience Publishers, Inc., New York, 1929. See especially pp. 347-354.

18. Kalaba, R., "On nonlinear differential equations, the maximum operation, and monotone convergence," J. of Math. and Mech., to appear.

19. Luneberg, R. K., "The propagation of electromagnetic plane waves in plane parallel layers," Research Report No. 172-3, New York University, Washington Square College, June, 1947.

20. Osterberg, H., "Propagation of plane electromagnetic waves in inhomogeneous media," J. Opt. Soc. America, vol. 48, 1958, pp. 513-521.

21. Pierce, J., "Note on the transmission line equations in terms of impedence," Bell Syst. Tech. J., vol. 22, 1943, pp. 263-265.

22. Redheffer, R., "Novel uses of functional equations," J. Rat. Mech. Anal., vol. 3, 1954, pp. 271-279.

23. Schelkunoff, S. A., "Remarks concerning wave propagation in stratified media," The Theory of Electromagnetic Waves, A Symposium, Interscience Publishers, Inc., New York, 1951, pp. 117-128.

24. Tatarskii, V. J., "On amplitude and phase pulsations of a wave propagating in a slightly inhomogeneous medium," Doklady Akad. Nauk SSSR, vol. 107, 1956, pp. 245-248.

25. Villars, F., and V. F. Weisskopf, "On the scattering of radio waves by turbulent fluctuations of the atmosphere," Proc. IRE, vol. 43, 1955, pp. 1232-1239.

26. Walker, L. R., and N. Wax, "Non-uniform transmission lines and reflection coefficients," J. Appl. Physics, vol. 17, 1946, pp. 1043-1045.

27. Wheelon, A. D., and R. B. Muchmore, "Line of sight propagation pnehomena—II. Scattered components," Proc. IRE, vol. 43, 1955, pp. 1450-1458.